Exact solutions for cosmological perturbations from generalized gravity theories

Carlos González (UDA), José Belinchón (UDA),
Ramón Herrera (PUCV)

Field Equations. We consider the action

$$S = \int d^4x \sqrt{-g} \left[\frac{F(\varphi)}{2} R + \frac{\omega(\varphi)}{2} g^{\mu\nu} \varphi_{;\mu}\varphi_{;\nu} - V(\varphi) \right], \qquad (1)$$

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \qquad (2)$$

we use conformal time $\eta = \int a^{-1} dt$. By using this variable, background equations are

$$3\mathcal{H}^2 = \frac{\omega\varphi'^2}{2F} + \frac{a^2 V}{F} - 3\mathcal{H}\frac{F'}{F} \tag{3}$$

$$\mathcal{H}^2 - \mathcal{H}' = \frac{\omega\varphi'^2}{2F} + \frac{F''}{2F} - \frac{\mathcal{H}F'}{F} \tag{4}$$

$$\frac{1}{2\omega}(a^2 F_{,\varphi} R - \omega_{,\varphi}\varphi'^2 - 2a^2 V_{,\varphi}) - 2\mathcal{H}\varphi' - \varphi'' = 0, \tag{5}$$

here $\mathcal{H} \equiv a'/a$ and $R = \frac{6}{a^2}(\mathcal{H}^2 + \mathcal{H}')$.

Perturbations. We consider line element for both scalar and tensor perturbation

$$ds_s^2 = a^2\{(1+2\Phi)d\eta^2 - [(1-2\Psi)\delta_{ij}]dx^i dx^j\}, \quad (6)$$

$$ds_t^2 = a^2\{d\eta^2 - (\delta_{ij} - h_{ij})dx^i dx^j\}. \quad (7)$$

Considering the invariant comoving curvature perturbation $\mathcal{R}$ defined by

$$\mathcal{R} \equiv \Psi + \mathcal{H}\frac{\delta\varphi}{\varphi'}, \quad (8)$$

the first order perturbed equations can be written as

$$\frac{1}{a^3 Q_s}\frac{d}{dt}(a^3 Q_s \dot{\mathcal{R}}_\kappa) + \frac{k^2}{a^2}\mathcal{R}_\kappa = 0, \quad (9)$$

where

$$Q_s = \frac{\frac{3F'^2}{2F} + \omega\varphi'^2}{(\mathcal{H} + \frac{F'}{2F})^2}, \tag{10}$$

and κ is a comoving wavenumber.
Defining the new variables: $z_s = a\sqrt{Q_s}$ y $v_\kappa = a\mathcal{R}_\kappa$, eq. (9)take the form

$$v''_\kappa + (k^2 - \frac{z''_s}{z_s})v_\kappa = 0, \tag{11}$$

exact solution for: $\frac{z''_s}{z_s} \propto \eta^{-2}$ and $\frac{Z''_s}{Z_s} = \frac{c_1}{\eta} + \frac{c_2}{\eta^2}$.

with

$$\frac{z_s''}{z_s} = \mathcal{H}^2\{(1+\delta_s)(2-\epsilon+\delta_s)+\frac{\delta_s'}{\mathcal{H}})\}. \quad (12)$$

where

$$\delta_s \equiv \frac{Q_s'}{2\mathcal{H}Q_s} \quad (13)$$

and ϵ is a slow roll parameter given by

$$\epsilon \equiv 1-\frac{\mathcal{H}'}{\mathcal{H}^2}. \quad (14)$$

if ϵ and δ_s constants:

$$\frac{z_{s,t}''}{z_{s,t}} = \frac{\gamma_{s,t}(\varphi)}{\eta^2}$$

with$\gamma_{s,t} = \frac{(1+\delta_{s,t})(2-\epsilon+\delta_{s,t})}{(1-\epsilon)^2}$

We consider now tensor perturbations h_{ij}, since h_i^j satisfies the same form eq.(9), with $Q_t = F$. In this case we have $z_t = a\sqrt{F}$, from which

$$\frac{z_t''}{z_t} = \mathcal{H}' + \mathcal{H}^2 + \mathcal{H}\frac{F'}{F} - \frac{F'^2}{4F^2} + \frac{F''}{2F}. \qquad (15)$$

Exact solution. In this paper we shall focus on ansatz

$$z_s = \alpha\eta^q, \tag{16}$$

hence

$$\frac{z_s''}{z_s} = q(q-1)\eta^{-2}. \tag{17}$$

Now, from eqs. (4) and (10) we can eliminate the term $\omega\varphi'^2$, then we use expression $z = a\sqrt{Q_s}$ and eq.(16) to find

$$F'' = -\frac{\alpha^2 \eta^{2q}}{a^2}(\mathcal{H} + \frac{F'}{2F})^2 + \frac{3F'^2}{2F} + 2F(\mathcal{H}^2 - \mathcal{H}') + 2\mathcal{H}F', \quad (18)$$

from which we found the solution

$$F = \frac{F_0 \eta^{2q}}{a^2}, \quad (19)$$

with

$$\alpha^2 = 2F_0(1 + q^{-1}). \quad (20)$$

By using relation $Q_s = \frac{z_s^2}{a^2}$ together to eqs. (16)and (19) we obtain

$$Q_s = 2(1 + q^{-1})F. \tag{21}$$

Now we can see eqs. (19) and (21) make eqs. (12) and (15) verify eq. (17), thus, we conclude

$$\frac{z_s''}{z_s} = \gamma_s \eta^{-2}, \tag{22}$$

$$\frac{z_t''}{z_t} = \gamma_t \eta^{-2}, \tag{23}$$

with $\gamma_s = \gamma_t = q(q-1)$.

Using eqs. (22) and (23) one solves eq. (11). The solutions for both cases for long wavelength limit are

$$v_\kappa = -\frac{\sqrt{\pi\eta}}{2}\frac{i}{\pi}\Gamma(\nu)\{\frac{\kappa|\eta|}{2}\}^{-\nu}, \tag{24}$$

where $\nu \equiv \sqrt{q(q-1)+1/4}$. Thus, the spectrum of scalar and tensor perturbation are given by

$$P_s = \frac{1}{Q_s}(\frac{H}{2\pi})^2(\frac{1}{aH|\eta|})^2[\frac{\Gamma(\nu)}{\Gamma(3/2)}]^2[\frac{\kappa|\eta|}{2}]^{3-2\nu}, \tag{25}$$

$$P_t = \frac{8}{Q_t}(\frac{H}{2\pi})^2(\frac{1}{aH|\eta|})^2[\frac{\Gamma(\nu)}{\Gamma(3/2)}]^2[\frac{\kappa|\eta|}{2}]^{3-2\nu}. \tag{26}$$

Since $Q_t = F$, we use the above equations and eq. (21) to find the tensor-scalar ratio r, we obtain

$$r \equiv \frac{P_t}{P_s} = 16(1 + 1/q). \tag{27}$$

and the spectral index is

$$n_s \equiv \frac{dlnP_s}{dln\kappa} = 4 - \sqrt{4q(q-1)+1}. \tag{28}$$

The potential. From eqs. (3),(4) and (19) we obtain

$$V = q(2q-1)\frac{F}{(a\eta)^2}. \tag{29}$$

Once we find the explicit function $\eta(\varphi)$, we shall use eq. (29) to establish the explicit form for $V(\varphi)$. It will be treated in the next section.

Inflationary solution. With eq.(19), eq. (4) becomes

$$\varphi'^2 = \frac{F}{\omega}\{-6\mathcal{H}^2 + 2q(1-2q)\eta^{-2} + 12q\mathcal{H}\eta^{-1}\}, \qquad (30)$$

where F is given by eq.(19) and ω is still free function.

Power law Case. An interesting case is $\omega = 1$ and $a \propto t^p$, that is, ($a \propto \eta^{\frac{p}{1-p}}$). Thus, with $F = F_0 \frac{\eta^{2q}}{a^2}$ eq. (30) gives

$$\varphi(\eta) = \alpha \eta^{\frac{p}{p-1}+q}, \tag{31}$$

with $\alpha = \frac{c\sqrt{F_0}}{\frac{p}{p-1}+q}$ and $c = -6(\frac{p}{p-1}) + 14q - 4q^2$. Thus we obtain

$$F = \frac{(\frac{p}{p-1} + q)^2}{c^2} \varphi^2 = \xi \varphi^2, \tag{32}$$

and the potential

$$V = \propto \varphi^{2\beta} \tag{33}$$

with $\beta = \frac{q-p(1+q)-1}{q-p(1+q)}$.

Exit problem. We choose $\varphi(\eta)$ via ansatz

$$\varphi'^2 = f\frac{F}{\omega}, \tag{34}$$

where f is a free function. there are three possibilities: $f = constant$,$f = f(\eta)$, $f = \mathcal{H}^2$ and $f = \mathcal{H}$.

Case $f = const \equiv A$. Inserting eq. (34) into eq.(30) with $A < 0$ and $\omega < 0$, we have the solutions

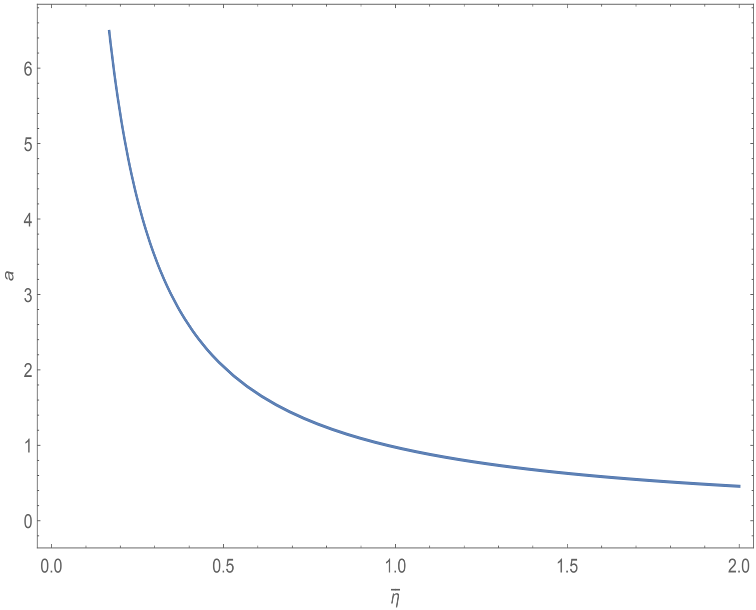
$$\mathcal{H}_1 = q\eta^{-1} - \sqrt{\frac{q(1+q)}{3}\eta^{-2} + \frac{|A|}{3}}, \tag{35}$$

$$\mathcal{H}_2 = q\eta^{-1} + \sqrt{\frac{q(1+q)}{3}\eta^{-2} + \frac{|A|}{3}}, \tag{36}$$

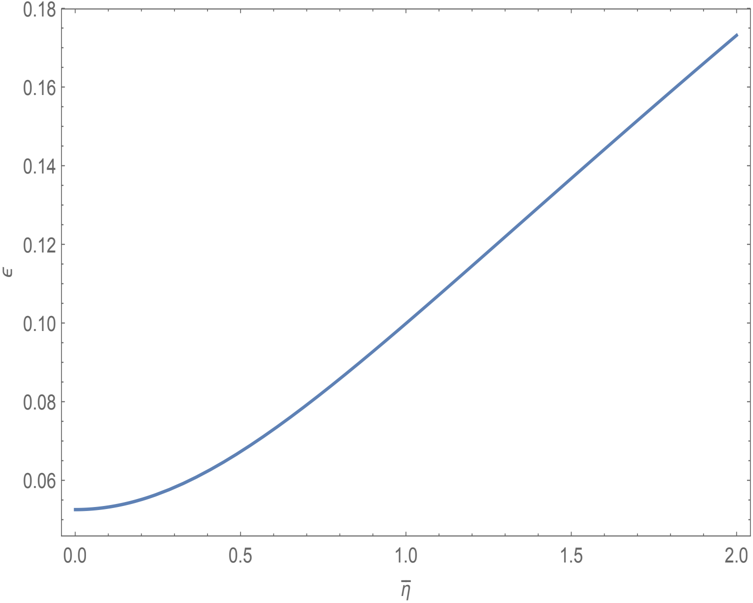
By choosing $\mathcal{H}_1$, we have

$$\bar{a} = \bar{\eta}^{q-\alpha}\{1 + \sqrt{1 + b^2\bar{\eta}^2}\}^{\alpha} e^{-\alpha\sqrt{1+b^2\bar{\eta}^2}}. \qquad (37)$$

Here we define $\bar{\eta} \equiv \frac{\eta}{\eta_i}$ and $\bar{a} = \frac{a}{a_i}$, where η_i and a_i represent conformal time and factor scale in the beginning of inflation respectively. The parameters b and α are defined by $b^2 \equiv \frac{|A|\eta_i^2}{2q(1+q)}$ and $\alpha \equiv \sqrt{\frac{q(1+q)}{3}}$.



6
5
4
3
2
1
0
a
0.0
0.5
1.0
1.5
2.0
$\bar{\eta}$


0.18
0.16
0.14
0.12
0.10
0.08
0.06
ϵ
0.0
0.5
1.0
1.5
2.0
$\bar{\eta}$

Eq.(34)is

$$\varphi(\eta) = \pm\sqrt{|A|}\int\sqrt{\frac{F}{|\omega|}}d\eta. \tag{38}$$

With (37) and (19) the above eq. becomes

$$\varphi(\eta) = \pm\sqrt{|A|F_0}\eta_i^q\int\frac{\bar{\eta}^\alpha e^{\alpha\sqrt{1+b^2\bar{\eta}^2}}}{\sqrt{|\omega|}\{1+\sqrt{1+b^2\bar{\eta}^2}\}^\alpha}d\eta. \tag{39}$$

Following the weak coupling limit of the low-energy effective string theory we have $\frac{F}{|\omega|} = \beta^2$, thus, choosing positive branch in (38) we have

$$\varphi(\eta) = \beta\sqrt{|A|}\eta + \varphi_0. \tag{40}$$

thus,

$$F(\varphi) \propto \{\frac{\varphi e^{\sqrt{1+c^2\varphi^2}}}{1+\sqrt{1+c^2\varphi^2}}\}^{2\alpha}, \tag{41}$$

and the potential

$$V(\varphi) \propto \varphi^{4\alpha-2-2q} e^{4\alpha\sqrt{1+c^2\varphi^2}} (1+\sqrt{1+c^2\varphi^2})^{-4\alpha}. \tag{42}$$

we note
for $c^2\varphi^2 << 1$ we have $F \to \varphi^{2\alpha}$, $V \to \varphi^{4\alpha-2-2q}$.
for $c^2\varphi^2 >> 1$ we have $F \to e^{-\varphi}$, $\omega \to -e^{-\varphi}$, with $(c < 0)$ and $V \to e^{-\varphi}\varphi^{-2-2q}$.

TO DO.

1.-To explore options:$f = f(\eta)$, $f = \mathcal{H}^2$ and $f = \mathcal{H}$.

2.-To study $\eta \to t$.

3.-Martin-Schwarz's exact solution: $\frac{Z''_{s,t}}{Z_{s,t}} = \frac{c_1}{\eta} + \frac{c_2}{\eta^2}$

GRACIAS