Classical duals to 3D gravity from a geometrical point of view

Patricio Salgado-Rebolledo
UAI

(In collaboration with G.Barnich and H. González [arXiv:1707.08887 [hep-th]])

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$$I_{EH} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R+\frac{2}{\ell^2}\right) = CS[A^+] - CS[A^-]$$

$$\boxed{CS[A] = \frac{\kappa}{4\pi}\int \mathrm{Tr}\left[AdA + \frac{2}{3}A^3\right] \qquad A^{a\,\pm} = \frac{e^a}{\ell} \pm \omega^a}$$

$$\text{HR} \longrightarrow \quad I_{cWZW} = \frac{\kappa}{4\pi}\int dtd\varphi\, Tr\left[g^{-1}\partial_\varphi g g^{-1}\partial_{\mp} g\right] - \kappa\Gamma\left[G\right]$$

$$\text{BC} \longrightarrow \quad I_{cb} = \frac{\kappa}{2\pi}\int dtd\varphi \partial_\varphi\phi\partial_{\mp}\phi$$

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$$\text{HR} \longrightarrow I_{flatWZW} = \frac{k}{\pi}\int dtd\phi Tr\left[\dot{\lambda}\lambda^{-1}\alpha' - \frac{1}{2}\left(\lambda'\lambda^{-1}\right)^2\right]$$

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- Asymptotic Symmetry= BMS_3

$$\begin{aligned} [\mathcal{J}_m, \mathcal{J}_n] &= (m-n)\mathcal{J}_{m+n} + \tfrac{c_1}{12}\delta_{m,-n} \\ [\mathcal{J}_m, \mathcal{P}_n] &= (m-n)\mathcal{P}_{m+n} + \tfrac{c_2}{12}\delta_{m,-n} \\ [\mathcal{P}_m, \mathcal{P}_n] &= 0 \end{aligned}$$

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- General solution of 3D Einstein eqs with $\Lambda < 0$ and Brown-Henneaux BC

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - (rdx^+ - \frac{8\pi G\ell}{r}b^- dx^-)(rdx^- - \frac{8\pi G\ell}{r}b^+ dx^+)$$

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- with $x^\pm = \frac{t}{\ell} \pm \varphi$ and the arbitrary 2π-periodic functions $b^\pm(x^\pm)$ transforming as

$$\tilde{b}^\pm = (\partial_\pm f^\pm)^2 b^\pm \circ f^\pm - c_\pm S_{x^\pm}[f^\pm]$$

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- Under the $\widehat{\mathrm{BMS}}_3$ transformations $p = p(\varphi)$ and $j = j(\varphi)$ transform as

$$\tilde{p} = (f')^2 p \circ f - \frac{c_1}{24\pi} S_\varphi[f]$$

$$\tilde{j} = (f')^2[j + \alpha p' + 2\alpha' p - \frac{c_2}{24\pi}\alpha'''] \circ f$$

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- Coadjoint action of the $\widehat{BMS}_3$ group [Barnich,Oblak (2015)]
- **Coadjoint orbits are endowed with a symplectic structure, which allows to construct geometric actions**

Outline

Adjoint action

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- A Lie group G can act on itself by conjugation

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- Infinitesimal adjoint action of $\mathfrak{g}$ on itself

$$ad_X Y = \left.\frac{d}{ds}\left(Ad_{g(s)} Y\right)\right|_{s=0} = [X, Y]$$

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- Differential of Ad^*_g at the identity: Infinitesimal coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$

$$\langle ad^*_X b, Y\rangle = -\langle b, ad_X Y\rangle$$

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- Coadjoint orbits are symplectic manifolds [Kirillov (1962), Kostant(1970)]

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- **Theorem: Pull-back of Ω on O_{b_0} defines a symplectic structure on O_{b_0}** [Kirillov (1974)]

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- **Theorem: Pull-back of Ω on O_{b_0} defines a symplectic structure on O_{b_0}** [Kirillov (1974)]
- Ω is closed $\Longrightarrow$ locally exact

$$\Omega = da, \quad a = \left\langle \mathrm{Ad}^*_{g^{-1}} b_0, \theta \right\rangle,$$

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- Form a representation of G under the action of the global symmetries,

$$\mathcal{L}_{v^R_{X_1}} Q_{X_2} = Q_{[X_1,X_2]}$$

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- $I_{\mathrm{G}} = \int_\gamma (a - H dt)$ preserves the symmetries

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- Ξ is a 2−cocycle on G

$$\Xi(g_1 g_2, g_3) + \Xi(g_1, g_2) = \Xi(g_1, g_2 g_3) + \Xi(g_2, g_3)$$

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- S is the Souriau cocycle on G

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- Infinitesimal adjoint action

$$\mathrm{ad}_{(X,n)}(Y,k) = [(X,n),(Y,k)] = ([X,Y], -\langle s(X), Y \rangle)$$

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- Generalized geometric action

$$I_{\mathrm{G}} = \int \left\langle \mathrm{Ad}^*_g b_0, \theta \right\rangle \longrightarrow I_{\hat{\mathrm{G}}} = \int \left\langle \mathrm{Ad}^*_{(g,m)} \left(b_0, c\right), \left(\theta, \theta_{\Xi}\right) \right\rangle$$

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- Maurer-Cartan equation

$$d\theta = -\frac{1}{2} \mathrm{ad}_\theta \theta \longrightarrow d(\theta, \theta_{\Xi}) = -\frac{1}{2} \mathrm{ad}_{(\theta, \theta_{\Xi})} (\theta, \theta_{\Xi})$$

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$$S\left(g\right)=\frac{1}{2\pi}g^{-1}\partial_\varphi g$$

- Maurer Cartan equation $d(\theta,\theta_\Xi)=\left(-\frac{1}{2}[\theta,\theta],\frac{1}{4\pi}\langle\partial_\varphi X,Y\rangle\right)$

$$(\theta,\theta_\Xi)=\left(g^{-1}dg,\ \frac{1}{4\pi}\left(\int_0^{2\pi}d\varphi\mathrm{Tr}\left[g^{-1}\partial_\varphi gg^{-1}dg\right]+\int_{\bar{D}}\mathrm{Tr}\left[g^{-1}\bar{d}gg^{-1}\bar{d}gg^{-1}dg\right]\right)\right)$$

Kač-Moody group

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- Centrally extended geometric action

$$I_{\widehat{\mathrm{LG}}}[g; b_0, c] = \int \left\langle \mathrm{Ad}^*_g b_0, \theta \right\rangle + c \int \left(- \left\langle S\left(g\right), \theta \right\rangle + \theta_{\Xi} \right)$$

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$$I_{\widehat{\mathrm{LG}}} = \int \int_0^{2\pi} d\varphi \mathrm{Tr}\left[b_0 dgg^{-1} - \frac{c}{4\pi} g^{-1}\partial_\varphi gg^{-1}dg \right] + c\,\Gamma[g]$$

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- Noether charges

$$Q_{(X,n)} = \int_0^{2\pi} d\varphi\, \mathrm{Tr}\left[\left(\frac{c}{2\pi} g^{-1}\partial_\varphi g - g^{-1} b_0 g\right) X(\varphi)\right]$$

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- The term proportional to b_0 can be absorbe by defining $u = \Upsilon(\varphi) g$ where $\Upsilon^{-1}\partial_\varphi \Upsilon = -\frac{2\pi}{c} b_0$

Kač-Moody group

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- Full geometric action

$$I_{\widehat{\mathrm{LG}}} = \frac{c}{2\pi}\int dtd\varphi \mathrm{Tr}\left[u^{-1}\partial_\varphi u u^{-1}\partial_- u\right] + c\,\Gamma[u]$$

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Chiral WZW model

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Chiral WZW model

- The dependence on the orbit representative b_0 is translated into a nontrivial periodicity of the field u,

$$u(\varphi+2\pi) = \mathcal{M}(b_0)u(\varphi), \quad \mathcal{M}(b_0) = \mathcal{P}\exp\left[-\frac{2\pi}{c}\oint d\varphi\, b_0\right]$$

Virasoro group

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- Diffeomorphism group of the circle $\mathrm{Diff}\left(S^1\right)$,

$$f(\varphi + 2\pi) = f(\varphi) + 2\pi, \quad f' > 0$$

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- Adjoint and coadjoint actions

$$\mathrm{Ad}_{f^{-1}}X=\frac{1}{f'(\varphi)}X(f(\varphi))\,\partial_\varphi \qquad \mathrm{Ad}^*_{f^{-1}}\,b=f'(\varphi)^2\;b(f(\varphi))\,(d\varphi)^2$$

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- Maurer-Cartan form

$$(\theta, \theta_\Xi) = \left(\frac{df}{f'}\partial_\varphi, \frac{1}{48\pi}\int_0^{2\pi} d\varphi \frac{df}{f'}\left(\frac{f''}{f'}\right)'\right)$$

Virasoro group

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$$H = \int d\varphi \left[b_0(f) f'^2 + \frac{c}{48\pi} \frac{f''^2}{f'^2} \right].$$

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- Defining $F = \Upsilon \circ f$ where Υ satisfies $c\, S[\Upsilon] = -b_0(\varphi)$ and $\partial_\varphi F = e^\chi$

$$\boxed{I_{\widehat{\mathrm{Diff}}(S^1)} = \frac{c}{24\pi} \int dt\, d\varphi\, \partial_\varphi \chi \partial_- \chi}$$

Chiral boson action

Semidirec products

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$$\langle (j, p)\,, (X, \alpha)\rangle_{\mathfrak{g}} = \langle j, X\rangle + \langle p, \alpha\rangle_{\mathcal{A}}$$

Semidirec products

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$$\begin{aligned} \mathrm{Ad}_{(g,\alpha)}(Y,\beta) &= \left(\mathrm{Ad}_g Y, \mathrm{Ad}_g \beta - \mathrm{ad}_{\mathrm{Ad}_g Y}\alpha\right) \\ \mathrm{Ad}^*_{(g,\alpha)^{-1}}(j,p) &= \left(\mathrm{Ad}^*_{g^{-1}} j - \mathrm{Ad}^*_{g^{-1}}\mathrm{ad}^*_{\alpha} p, \mathrm{Ad}^*_{g^{-1}} p\right) \end{aligned}$$

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- Geometric action

$$I_{\mathcal{S}_{\mathrm{Ad}}} = \int \left\langle \mathrm{Ad}^*_g \left(j_0 - \mathrm{ad}^*_\alpha p_0\right), \theta \right\rangle$$

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- Defining new fields $u = \Upsilon g$ and $a = \eta + \mathrm{Ad}_\Upsilon \alpha$ such that

$$c_2 S(\Upsilon) = -p_0, \quad c_2 \mathrm{Ad}^*_{\Upsilon^{-1}} s(\eta) = -j_0 + \frac{c_1}{c_2} p_0$$

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- Geometric action on a coadjoint orbit $O_{(j_0, c_1, p_0, c_2)}$

$$I_{\widehat{\mathcal{S}}_{\mathrm{Ad}}} = c_1 \int \left(-\langle S(u), \theta \rangle + \theta_\Xi \right) - c_2 \int \langle \mathrm{Ad}^*_u (s(a)), \theta \rangle$$

Kač-Moody case

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$$\mathcal{J}_X = \int_0^{2\pi} d\varphi\, \mathrm{Tr}\,[Xj]\,, \qquad j = \frac{c_1}{2\pi} g^{-1}\partial_\varphi g - g^{-1}\left(j_0 - \frac{c_2}{2\pi}\partial_\varphi \alpha - [\alpha, p_0]\right) g$$

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I_{\widehat{LG}\ltimes_{\mathrm{Ad}}\widehat{L\mathfrak{g}}_{\mathrm{ab}}} &= \frac{c_1}{2\pi}\int dt d\varphi \mathrm{Tr}\left[u^{-1}\partial_\varphi u u^{-1}\partial_- u\right] + c_1\,\Gamma[u] \\
&- \frac{c_2}{2\pi}\int d\varphi dt \mathrm{Tr}\left[\dot{u}u^{-1}a' - \frac{1}{2}(u^{-1}u')^2\right]
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- Classical dual to 3D flat gravity (with non-periodic fields)

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Thank you!