Renyi entropies of spinor fields via AdS/CFT

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Introduction

- The study of entropy for physical systems has had many surprises in the last 50 years.
- "Classical" expected value proportional to volume.
- Bekenstein - Hawking entropy $\rightarrow$ proportional to area of black hole horizon.
- Are we talking about the "same" entropy?

Introduction

- (1985) 't Hooft: entropy of thermal gas of Hawking particles outside of BH horizon (brick-wall).
- (1993) Srednicki numerically calculated entropy inside of a imaginary surface (*entanglement entropy*).
  1. Entanglement entropy depends only of states near the entangling surfaces.
  2. Entanglement entropy is UV sensitive.
  3. Not necessary a BH for area law.

  EE = von Neumann entropy of the reduced density matrix

$$\begin{aligned} \rho_A &= \mathrm{tr}_B\{\rho_{A\oplus B}\} \\ S_{EE} &= -\mathrm{tr}_A\{\rho_A \log \rho_A\} \end{aligned} \tag{1}$$

- For free fields in d+1 dimensions with spherical entangling surface

$$S_{EE} \sim \frac{\text{Area of entangling surface}}{\epsilon^{d-1}} \tag{2}$$

Introduction

- 2-dimensional case CFT:

$$S_{EE} = -\frac{c}{3}\log\epsilon \tag{3}$$

- Arbitrary even dimensional case

$$S_{EE} = \frac{g_{d-2}}{\epsilon^{d-2}} + ... + \frac{g_2}{\epsilon^2} + g_0 \log\epsilon + S_0 \tag{4}$$

  g_i are local functions of edge surface ∂V and S_0 is the finite part.

  H. Casini and M. Huerta.

- In even d+1 dimensions, logarithmic term on spheres is related to Type-A trace anomaly a.

Introduction

- Casini & Huerta related EE (for conformal scalar) to thermal entropy in $S^1 \times H^d$ via conformal maps.

$$Z_{S^1 \times H^d} = \det Y_{S \times H} \tag{5}$$

- Idea! Use holographic formula to compute these determinants in gravity.
- Is it possible, by modifying some parameter, to compute Renyi entropies along these lines?
- Renyi entropies $\rightarrow$ a anomaly.

Conformal scalar field

Entanglement entropy

- From Casini & Huerta: EE is related to thermal entropy

$$\det Y_{S^d} \tag{6}$$

- From Holographic formula:

$$\left.\frac{\det_-(\nabla^2 + m_k^2)}{\det_+(\nabla^2 + m_k^2)}\right|_{H^{d+1}} = \det Y\big|_{S^d} \tag{7}$$

m_k are special value of mass related to conformal field in the boundary.

- Using heat kernel method

$$-\frac{1}{2}\log\det P_{2k}\Big|_{S^d} = \frac{-\mathrm{vol}(H^{d+1})}{(4\pi)^{d/2+1/2}}\sum_{n=0}^{d/2-1} a_n^{(d+1)}\Gamma(n-1/2-d/2)k^{d+1-2n} \tag{8}$$

Conformal scalar field

Entanglement entropy

- D.E. Diaz (2008), J.S. Dowker(2011)

$$-\frac{1}{2}\log\det P_{2k}\Big|_{S^d} = \mathrm{vol}(H^{d+1})\int_0^k \frac{1}{(4\pi)^{d/2}}\frac{(\nu)_{d/2}(-\nu)_{d/2}}{(1/2)_{d/2}} \quad (9)$$

- How to formally obtain the analytic result by Casini & Huerta?
  1. First compute
  $$-2\frac{(-1)^d/2}{(d-2)!}\left[\frac{d}{dz}\right]_{q^{-1}}\left\{z\int_0^z d\nu(\nu)_{d/2-1}(-\nu)_{d/2-1}\right\} \quad (10)$$
  q-derivative is defined by
  $$\left[\frac{df}{dz}\right]_q = \frac{f(qx)-f(x)}{qx-x} \quad (11)$$
  2. Then replace powers of zeta with Riemann zetas
  $$z^l \to \xi[-l] \quad (12)$$
  3. Value at $q = 1$.

Conformal scalar field

Renyi entropy

- Holographic formula can be used for EE within $S^1(\beta) \times H^{d-1}$.
- We can modify temperature parameter $2\pi \to 2\pi q$.
- For $q = 1 \to$ Entanglement entropy.
- $q \neq 1 \sim$ introducing angular deficit in bulk interior.
- Renyi entropy defined by

$$S_q = \frac{\log Z(2\pi q) - q \log Z(2\pi)}{1 - q} \tag{13}$$

- Using Sommerfeld's formula for heat kernel we can obtain the known results for conformal scalar in even dimensions.

Massless Dirac field

Renyi entropy

- Our idea is use the same method to find Renyi entropies for massless Dirac field.
- Bulk metric:

$$\begin{aligned} ds^2_{H^{d+1}} &= d\mu^2 + \sinh^2\mu d\tau^2 + \cosh^2\mu ds^2_{H^{d-1}} \quad (14) \\ \tau &\sim \tau + 2\pi q \quad (15) \end{aligned}$$

- As conformal scalar: Dirac field is massive in gravity side → need to compute

$$\frac{\det_-(\not\nabla + 1/2)}{\det_+(\not\nabla + 1/2)} \quad (16)$$

Massless Dirac field

Renyi entropy

- Heat kernel depends on geodesic distance σ
  R. Camporesi, The spinor heat kernel in maximally symmetric spaces

$$K_d(x,t) = U(y)\cosh(\frac{\sigma}{2})\left(\frac{-1}{2\pi}\frac{\partial}{\partial\cosh\sigma}\right)^{\frac{d-1}{2}} \times \left(\cosh(\frac{\sigma}{2})\right)^{-1}\frac{e^{-\sigma^2/4t}}{(4\pi t)^{1/2}} \quad (17)$$

- Here we recall the Sommerfeld's formula for spinor

$$K^*(\mu,\tau-\tau';t) = K(\mu,\tau-\tau';t) - \frac{i}{4\pi q}\int_C dw\,\csc\left(\frac{w}{2q}\right)K(\mu,\tau-\tau'+w;t) \quad (18)$$

- w (angular parameter) and μ are related by

$$\sinh\frac{\sigma}{2} = \sin\frac{w}{2}\sinh\mu \quad (19)$$

Massless Dirac field

Renyi entropy

- To obtain the determinant is necessary to take trace of heat kernel generating a volume divergence times a residue which easily can be computed (by Maple, obviously).
- Example: d=2

$$
\begin{aligned}
\mathrm{tr}\int_0^\infty K_3^*(\sigma;t) &= \frac{\mathrm{vol}(H^1)}{2i}\int_C dw \csc\left(\frac{w}{2q}\right)\int_0^\infty \frac{dt}{t} \\
&\times \int_0^\infty d\mu \sinh\mu \cosh\mu K_3(\sigma;t) \qquad (20)
\end{aligned}
$$

- After computing residue:

$$
g_0^q = -\frac{1+q}{6q} \qquad (21)
$$

valid for any field in 2 dimensions.

Massless Dirac field

Renyi entropy

- d=4 case:

$$\begin{aligned} \text{tr} \int_0^\infty K_5^*(\sigma; t) &= \frac{\text{vol}(H^3)}{2i} \int_C dw \csc\left(\frac{w}{2q}\right) \int_0^\infty \frac{dt}{t} \\ &\times \int_0^\infty d\mu \sinh\mu \cosh^3\mu K_5(\sigma; t) \end{aligned} \quad (22)$$

- Again, after computing residue we got:

$$g_0^q = -\frac{(1+q)(7+37q^2)}{720q^3} \quad (23)$$

- It is possible to construct a recurrence formula for higher dimensional case

Massless Dirac field

Renyi entropy

- d=6 case:

$$g_0^q = \frac{(1+q)(31+276q^2+1221q^4)}{60480q^5} \tag{24}$$

- d=8 case:

$$g_0^q = -\frac{(1+q)(381+4721q^2+30103q^4+124603q^6)}{14515200q^7} \tag{25}$$

J. Lee, A. Lewkowycz, E. Perlmutter, B. Safdi, "Renyi entropy, stationarity, and entanglement of the conformal scalar"

Massless Dirac field
Renyi entropy

- Is there a simpler recipe?
- As in the scalar case, there is a q- analog recipe:
  1. First compute the polynomial

$$\frac{(-1)^{d/2}2^{1+d/2}}{(d-2)!}\left[\frac{d}{dz}\right]_{q^{-1}}\left\{z\int_0^z d\nu(\frac{1}{2}+\nu)_{\frac{d}{2}-1}(\frac{1}{2}-\nu)_{\frac{d}{2}-1}\right\} \quad (26)$$

  2. Replace powers of zeta by Hurwitz zeta functions:

$$z^l \to \zeta[-l,\frac{1}{2}] \quad (27)$$

Massless Dirac field

Comments in d=4

- We can *rename* g_0^q as $f_a(q) \rightarrow$ trace anomaly coefficient a.
- There is another trace anomaly coefficient c $\rightarrow f_c(q)$.
- In principle there is no relation between $f_c(q)$ and $f_a(q)$.
- Conjecture:

$$c = f_c(q = 1) = -2f_a'(1) \tag{28}$$

for any CFT. (A. Lewkowycz and E. Perlmutter)

- Checked for scalar and spinor cases.

Conclusions and outlook

- Holographic formula holds for conical singularity.
- Renyi entropies can be computed using gravity dual of conformal field in $S(2\pi q) \times H^d$.
- Anomaly coefficients for spinor cases can be obtained via AdS/CFT.
- Problem with vector case?
- Conformal Higher Spin fields (CHS): $f_c(q)$ or $f_a(q)$??
- Can be possible to write a q-recipe for CHS?

Many thanks!