On the EINSTEIN/CONFORMAL GRAVITY equivalence:

role of the Q-curvature

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acknowledge useful conversations with R.Aros, F.Bugini, R.Olea,...

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Aim of the talk

- Question: (physically motivated)

  In even dimensions, what are the conformal gravities admitting all Einstein metrics as solutions?

- Answer: (already known to conformal geometers)

  Branson´s Q-CURVATURE

- Question: What is Q-CURVATURE?

Outline

Preliminaries

4D $\to$ 6D

Even D: Einstein Gravity and Q-curvature

Outlook

Modifications of Einstein gravity

- What modifications?... After 100 years, there are plenty of proposals (vs. observations/experiments).

- In this talk: higher-curvature corrections

  Naturally appear in the search for quantum UV completion (e.g. string theory) where GR turns up as low-energy effective field theory description.

  Pros: may render GR perturbatively renormalizable [K.Stelle'77–78]

$$\int_{M_4} R + \alpha\, Ric^2 + \beta\, R^2$$

  Cons: clash with unitarity (ghosts)

$$\frac{M^2}{\Box(\Box + M^2)} = \frac{1}{\Box} - \frac{1}{\Box + M^2}$$

4D: Critical Gravity

Prompted by constructions in 3D massive gravities (‘topological’, ‘new’, ‘chiral’, ...), consider Cosmological Einstein-Weyl Gravity

$$I_{EW} = \int_{M_4} R - 2\Lambda + \frac{1}{2}\alpha\, Weyl^2$$

Surprises at the *critical* point $\alpha = 3/2\Lambda = -1/2$ [H.Lü+C.Pope’11]

- Massive spin-2 field becomes massless
- Traded by log-modes (ghost-like but non-tachyonic) with slower fall-off than massless modes, so they can be truncated out by appropriate AdS boundary conditions
- But a bunch of null results: energy of massless gravitons, mass and entropy of Schwarzschild-AdS black holes

NB: A possible way out, in analogy with Breitenlohner-Freedman window for scalars.

4D: Weyl and Lanczos Gravities

Historical remark

- Weyl-squared gravity [R.Bach'21]

$$\int_{M_4} Weyl^2$$

- Modulo Chern-Gauss-Bonnet integrand [K.Lanczos'38]

$$2\int_{M_4} Ric^2 - \frac{1}{3}R^2$$

- Metric variation: vanishing Bach tensor ('pure Ricci')

$$B_{ij} = (\nabla^k \nabla^l - \frac{1}{2}R^{kl})\, W_{ikjl}$$

conformally Einstein $\subset$ Bach-flat

4D: Einstein/Conformal Gravity equivalence

4D conformal gravity revisited: two observations [J.Maldacena'11]

- Within Bach-flat, Einstein metrics are chosen by Neumann boundary condition in the Fefferman-Graham expansion
- Euclidean Renormalized CEH = Weyl-squared action [M.Anderson'00] [O.Miskovic+R.Olea'09]

$$6\,\widehat{V}(M_4) \quad + \frac{1}{4}\int_{M_4} Weyl^2 = 8\pi^2\,\chi(M_4)$$

$$^{ren}I_{CEH} \quad -\frac{1}{4}\int_{M_4} Weyl^2 = -8\pi^2\,\chi(M_4)$$

★ Remarkable: the very same coefficient as in Critical Gravity

∴ Null results in Critical Gravity then follow from this on-shell Einstein/Weyl equivalence

NB: A related 'generalized Chern-Gauss-Bonnet' was first noticed in Valdivia!

[R.Aros+M.Contreras+R.Olea+R.Troncoso+J.Zanelli'99]

6D: Einstein/Conformal Gravity equivalence

Why 6D?

- Strings, extra dimensions, brane-world scenarios.
- KK compactifications result in effective 4D conformal gravity plus other fields.
- Discern between features that are specific to 4D and *universal* ones.

Proper extension of Einstein/Conformal Gravity equivalence [H.Lü+Y.Pang+C.Pope'12]

1. Which 6D conformal gravity?

   3 'games in town' $W_{ijkl}W^{klmn}W^{ij}{}_{mn}$ $\quad W_{ikjl}W^{kmln}W_{m}{}^{i}{}_{n}{}^{j}$ $\quad W_{ijkl}\Box W^{ijkl} + \ldots.$

   The combination that has all Einstein metrics as solutions: the 'pure Ricci' one found in the holographic trace anomaly computation [M.Henningson+K.Skenderis'98]

2. What about the renormalized actions?

   There is a corresponding generalized Chern-Gauss-Bonnet formula [A.Chang+J.Qing+P.Yang'05, P.Albin'05] [O.Miskovic+R.Olea+M.Tsoukalas'14]

Even D

Proper extension of Einstein/Conformal Gravity equivalence

1. The combination that has all Einstein metrics as solutions: the 'pure Ricci' combination in the holographic trace anomaly

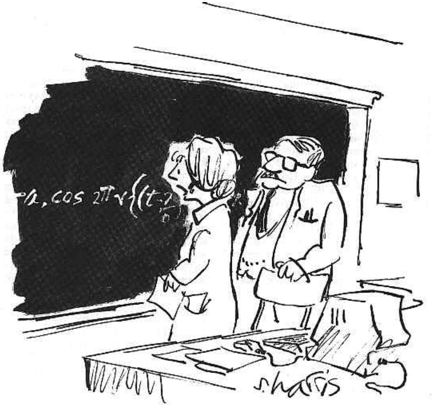
   Theo [R.C.Graham+M.Zworski'01]: holographic trace anomaly is the integral of Branson's Q-curvature

   $$\int_{M^n} Q_n$$

   Theo [R.C.Graham+K.Hirachi'04]: the metric variation of Q-curvature is the Fefferman-Graham obstruction tensor O_{ij}, and there is no obstruction for conformally Einstein metrics

2. What about the renormalized actions?

   There is a corresponding generalized Chern-Gauss-Bonnet formula
   [A.Chang+J.Qing+P.Yang'05, P.Albin'05] [O.Miskovic+R.Olea+M.Tsoukalas'14]



"IT'S AN EXCELLENT PROOF, BUT IT LACKS WARMTH AND FEELING."

Fefferman-Graham program

C.Fefferman+R.C.Graham, 1984 :

- Embedding $(\mathcal{M}^n, [g])$ in a Ricci-flat ambient metric in two dimensions higher
- A workhorse at work: ambient Riemannian invariants ‘induce’ point-wise conformal invariants
- In odd dimensions: all !
- In even dimensions: not all, but enough to built those of weight -n
- The procedure also produces conformally covariant differential operators, e.g. GJMS operators (conformal powers of the Laplacian)

$$P_n^{(m)} = \Delta^m + LOT$$

Branson's Q-curvature

- Under Weyl rescaling of the metric $g \to \hat{g} = e^{n\,w}$ in n=even:

$$e^{nw}\,\hat{Q}_n \;=\; Q_n \,+\, P_n\,w$$

- Examples of Q-curvatures (modulo overall numerical factors)

$$n = 2: \;\; Q_2 = R \qquad and \qquad P_2 = \Delta$$

$$n = 4: \;\; Q_4 = E_4 - W^2 + \frac{1}{3}\Delta R \quad and \quad P_4 = \Delta^2 + LOT$$

NB: P_4 is called Paneitz operator, but in fact it was discovered by E.Fradkin and A.Tseytlin (Conformal SUGRA papers of 1982)

Some facts about Q_{even}

- In the FG construction, there is also a Poincaré metric (LAAdS?) so that its conformal infinity (boundary) is $(\mathcal{M}^n, [g])$. A representative g in the conformal class $[g]$ is associated to a defining function x

$$g_+ = x^{-2}\left(dx^2 + h(x)\right) \qquad with\, h(0) = g$$

- Prompted by AdS/CFT (E.Witten, Henningson+K.Skenderis), compute the renormalized volume (or action for EH gravity in the bulk) (C.R.Graham, P.Albin)

$$Vol_{g_+}(\{x > \epsilon\}) = \frac{c_0}{\epsilon^n} + \frac{c_2}{\epsilon^{n-2}} + (even\, powers) + L\, log\frac{1}{\epsilon} + V + o(1)$$

- L (integrated holographic trace anomaly!) is conformal invariant and, remarkably (C.R.Graham+M.Zworski)

$$L = \int_{\mathcal{M}^n} Q_n\, dvol(g)$$

Outlook

Not alone! Lots of interesting parallel (and potentially useful) developments!

For some thoughts on 4D conformal gravity refer to (please google them)

- P.Mannheim et al.
- G. ’t Hooft
- Scattering amplitudes and twistors ... and all that

MANY THANKS