Some simple exact solutions to $d = 5$ Einstein–Gauss–Bonnet Gravity

Eduardo Rodríguez

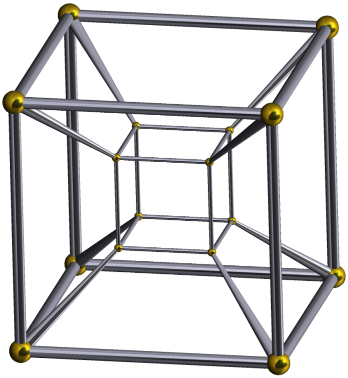
Departamento de Matemática y Física Aplicadas
Universidad Católica de la Santísima Concepción
Concepción, Chile

CosmoConce, UBB, Concepción, Chile
March 15–16, 2012

The Question we would like to address

(Not answer, of course)

Why Four?



The Question we would like to address

To answer the question, we should begin by...

- We will never answer *Why Four?* if we assume from the outset that spacetime is four-dimensional
- Higher-dimensional theories (e.g. String Theory, Supergravity) usually assume some kind of compactification
- Can we get an effectively four-dimensional spacetime emerging from a higher-dimensional theory?

Einstein–Gauss–Bonnet Gravity

Usual Tensor Formulation

$d = 5$ EGB Action in tensor notation

$$S^{(5)}_{\mathrm{EGB}} = \int_M d^5x \sqrt{-g}\,[\gamma_0 + \gamma_1 R + \\ +\gamma_2 \left(R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\mu\nu} - 4 R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} + R^2 \right)]$$

$d = 5$ EGB Action explained

The action includes three terms:

- a cosmological constant term
- the usual Einstein–Hilbert term
- a curvature-squared “Gauss–Bonnet” term

Einstein–Gauss–Bonnet Gravity

Usual Tensor Formulation

$d = 5$ EGB Action in tensor notation

$$S^{(5)}_{\mathrm{EGB}} = \int_M d^5x \sqrt{-g} \left[\gamma_0 + \gamma_1 R + \right.$$
$$\left. + \gamma_2 \left(R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\mu\nu} - 4 R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} + R^2 \right) \right]$$

$d = 5$ EGB Action explained

The action includes three terms:

- a cosmological constant term
- the usual Einstein–Hilbert term
- a curvature-squared "Gauss–Bonnet" term

Einstein–Gauss–Bonnet Gravity

Usual Tensor Formulation

$d = 5$ EGB Action in tensor notation

$$S^{(5)}_{\mathrm{EGB}} = \int_M d^5x\sqrt{-g}\,[\gamma_0 + \gamma_1 R + \\ + \gamma_2 \left(R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\mu\nu} - 4R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} + R^2 \right)]$$

$d = 5$ EGB Action explained

The action includes three terms:

- a cosmological constant term
- the usual Einstein–Hilbert term
- a curvature-squared "Gauss–Bonnet" term

Einstein–Gauss–Bonnet Gravity

Usual Tensor Formulation

$d = 5$ EGB Action in tensor notation

$$S^{(5)}_{\text{EGB}} = \int_M d^5x\sqrt{-g}\,[\gamma_0 + \gamma_1 R + \\ +\gamma_2 \left(R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu} - 4R^{\mu}{}_{\nu}R^{\nu}{}_{\mu} + R^2\right)]$$

$d = 5$ EGB Action explained

The action includes three terms:

- a cosmological constant term
- the usual Einstein–Hilbert term
- a curvature-squared “Gauss–Bonnet” term

Einstein–Gauss–Bonnet Gravity

First-order Formulation

$d = 5$ EGB Lagrangian in first-order formulation

$$L_{\text{EGB}}^{(5)} = \epsilon_{abcde}\left(\alpha_0 \mathrm{e}^a\mathrm{e}^b\mathrm{e}^c\mathrm{e}^d\mathrm{e}^e + \alpha_1 R^{ab}\mathrm{e}^c\mathrm{e}^d\mathrm{e}^e + \alpha_2 R^{ab}R^{cd}\mathrm{e}^e\right).$$

Field Content

- $\mathrm{e}^a = \mathrm{e}^a{}_\mu \mathrm{d}x^\mu$: vielbein
- $\omega^{ab} = \omega^{ab}{}_\mu \mathrm{d}x^\mu$: spin connection
- $R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c\omega^{cb}$: Lorentz curvature
- $T^a = \mathrm{d}\mathrm{e}^a + \omega^a{}_b\mathrm{e}^b$: Torsion

An Open Problem in $d = 5$ EGB Gravity

What is the vacuum of the theory?

Field Equations for the EGB Theory

$$\epsilon_{abcde}\left(5\alpha_0 e^a e^b e^c e^d + 3\alpha_1 R^{ab} e^c e^d + \alpha_2 R^{ab} R^{cd}\right) = 0,$$

$$\epsilon_{abcde}\left(3\alpha_1 e^c e^d + 2\alpha_2 R^{cd}\right) T^e = 0.$$

Factorizing the Field Equations

$$\beta_0 \epsilon_{abcde}\left(R^{ab} - \beta_1 e^a e^b\right)\left(R^{cd} - \beta_2 e^c e^d\right) = 0.$$

Relation between the α's and the β's

$$5\alpha_0 + 3\alpha_1 x + \alpha_2 x^2 = \beta_0 \left(x - \beta_1\right)\left(x - \beta_2\right)$$

Selecting the Coefficients

The vacuum structure of the EGB theory depends strongly on the α's

- When β_1 and β_2 are real and distinct, there are *two* vacuum states with constant curvature.
- When $\beta_1 = \beta_2$ then there is a single vacuum state with constant curvature. This is a special case, because the action acquires a larger symmetry for this choice of coefficients and becomes the Chern–Simons action for the (A)dS group.
- What happens when the β_i are complex?

Selecting the Coefficients

The vacuum structure of the EGB theory depends strongly on the α's

- When β_1 and β_2 are real and distinct, there are *two* vacuum states with constant curvature.
- When $\beta_1 = \beta_2$ then there is a single vacuum state with constant curvature. This is a special case, because the action acquires a larger symmetry for this choice of coefficients and becomes the Chern–Simons action for the (A)dS group.
- What happens when the β_i are complex?

Selecting the Coefficients

The vacuum structure of the EGB theory depends strongly on the α's

- When β_1 and β_2 are real and distinct, there are *two* vacuum states with constant curvature.
- When $\beta_1 = \beta_2$ then there is a single vacuum state with constant curvature. This is a special case, because the action acquires a larger symmetry for this choice of coefficients and becomes the Chern–Simons action for the (A)dS group.
- What happens when the β_i are complex?

One theory, three different regimes

Vacuum structure of the EGB theory parameterized by single constant χ

- It is convenient to parameterize the Lagrangian as

$$L^{(5)}_{\mathrm{EGB}} = \frac{\kappa}{l}\epsilon_{abcde}\left(R^{ab}R^{cd} - \frac{2\chi}{3l^2}R^{ab}e^c e^d + \frac{1}{5l^4}e^a e^b e^c e^d\right)e^e$$

- $\chi^2 > 1$: two constant-curvature vacua
- $\chi^2 = 1$: one constant-curvature vacuum
- $\chi^2 < 1$: no constant-curvature vacua
- Why may be this last, “pathological” case be interesting?

Warped Product Ansatz

Warped product solutions as a means towards dynamical dimensional reduction

General Warped Product Ansatz

$$ds^2 = -f^2(w)\,dt^2 + g^2(w)\,d\Sigma^2 + p^2(t)\,q^2(x,y,z)\,dw^2,$$

where Σ is a constant-curvature 3-manifold:

$$d\Sigma^2 = \left[1 + \frac{K}{4}\left(x^2 + y^2 + z^2\right)\right]^{-2}\left(dx^2 + dy^2 + dz^2\right).$$

Warped Product Solutions

Plugging the ansatz in the field equations we find...

The field equations imply the following:

$$g(w) = 1,$$
$$q(x, y, z) = 1,$$
$$K = \frac{1}{\chi l^2}.$$

Simplified Ansatz

$$ds^2 = -f^2(w)\,dt^2 + d\Sigma^2 + p^2(t)\,dw^2.$$

Summary of Solutions

Different solutions for the EGB theory with $\chi^2 < 1$

Class	$f(w)$	$p(t)$	Σ	χ-range	R, τ
PH$-$	1	hyp.	$K < 0$	$-1 < \chi < 0$	$l\sqrt{\xi}$
FC$-$	circ.	1	$K < 0$	$-1 < \chi < 0$	$l\sqrt{\xi}$
PC$+$	1	circ.	$K > 0$	$0 < \chi < 1$	$l\sqrt{-\xi}$
FH$+$	hyp.	1	$K > 0$	$0 < \chi < 1$	$l\sqrt{-\xi}$

$$\xi = \frac{1}{2}\left(\chi - \frac{1}{\chi}\right)$$

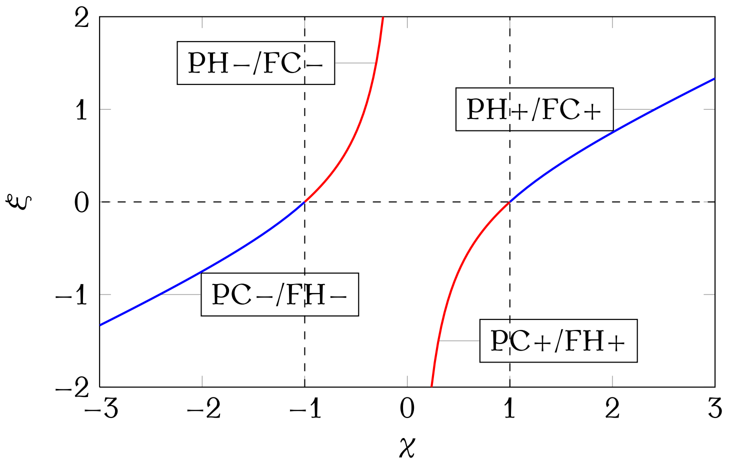
Summary of Solutions

Different solutions for the EGB theory with $\chi^2 > 1$

Class	$f(w)$	$p(t)$	Σ	Theory	Range	R, τ	K
PH+	1	hyp.	$K > 0$	EGB	$\chi > 1$	$l\sqrt{\xi}$	$\frac{1}{\chi l^2}$
				GR	$\Lambda > 0$	$\sqrt{\frac{3}{2\Lambda}}$	$\frac{\Lambda}{3}$
FC+	circ.	1	$K > 0$	EGB	$\chi > 1$	$l\sqrt{\xi}$	$\frac{1}{\chi l^2}$
				GR	$\Lambda > 0$	$\sqrt{\frac{3}{2\Lambda}}$	$\frac{\Lambda}{3}$
PC−	1	circ.	$K < 0$	EGB	$\chi < -1$	$l\sqrt{-\xi}$	$\frac{1}{\chi l^2}$
				GR	$\Lambda < 0$	$\sqrt{-\frac{3}{2\Lambda}}$	$\frac{\Lambda}{3}$
FH−	hyp.	1	$K < 0$	EGB	$\chi < -1$	$l\sqrt{-\xi}$	$\frac{1}{\chi l^2}$
				GR	$\Lambda < 0$	$\sqrt{-\frac{3}{2\Lambda}}$	$\frac{\Lambda}{3}$

Summary of Solutions

A graphic summary of solutions and theories



Circular Fifth Dimension

An FC− solution features a circular fifth dimension

Line element for the FC− spacetime

$$ds^2 = -\cos^2\left(\frac{w}{R}\right) dt^2 + d\Sigma^2 + dw^2$$

Features

- Compact (circular) fifth dimension of radius R
- Flow of time changes along the circle

Circular Fifth Dimension

An PH− solution features a shrinking fifth dimension

Line element for the PH− spacetime

$$ds^2 = -dt^2 + d\Sigma^2 + e^{-2t/\tau} dw^2$$

Features

- Fifth dimension shrinks exponentially
- Effectively four-dimensional spacetime emerges dynamically after some time τ

Some Open Problems

Or where this road might lead us next

- Are the solutions *stable*?
- Is this the vacuum for the EGB theory with $\chi^2 < 1$?
- What happens in higher dimensions?
- Can we include matter and nontrivial torsion?