Fermionic and scalar fields as sources of interacting dark matter-dark energy

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Model: Fermions $\psi : \mathcal{M} \to \mathbb{R}^4$ and scalars $\phi : \mathcal{M} \to \mathbb{R}$ with self-interacting potential density $V(\bar{\psi}, \psi) : \mathcal{M} \to \mathbb{R}$ and $U(\phi) : \mathcal{M} \to \mathbb{R}$ in presence of a gravitational field. Dynamics between fermions and scalars fields are represented Yukawa type interaction $f(\phi) : \mathcal{M} \to \mathbb{R}$.
The action is,

$$\mathcal{S}(g, \psi, \bar{\psi}, \phi) = \int d^4x \sqrt{-g}\mathcal{L}, \tag{1}$$

where

$$\mathcal{L} = \frac{1}{2}(1 - \xi\phi^2)R + \frac{i}{2}\left(\bar{\psi}\Gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\Gamma^\mu \psi\right) - m\bar{\psi}\psi + -V(\bar{\psi}, \psi) + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi) - \bar{\psi} f(\phi)\psi, \tag{2}$$

spinors are treated as classical commuting fields and $\xi \in \mathbb{R}$ is a non-minimal coupling between gravitation and scalar fields.

We are using natural units $8\pi G = c = \hbar = 1$. Dirac matrices in curved space are defined by

$$\Gamma^{\mu} = e^{\mu}_{a}\gamma^{a}; \ \ \{\Gamma^{\mu}, \Gamma^{\nu}\} = 2g^{\mu\nu}$$

. The covariant derivative is defined

$$\begin{aligned}
\nabla_{\mu}\psi &= \partial_{\mu}\psi - \Omega_{\mu}\psi, \\
\nabla_{\mu}\bar{\psi} &= \partial_{\mu}\bar{\psi} + \bar{\psi}\Omega_{\mu}, \\
\Omega_{\mu} &= -\frac{1}{4}g_{\mu\nu}\left[\Gamma^{\nu}_{\sigma\lambda} - e^{\nu}_{b}(\partial_{\sigma}e^{b}_{\lambda})\right]\gamma^{\sigma}\gamma^{\lambda}
\end{aligned}$$

Defining $\alpha \equiv 1 - \xi\phi^{2}$ it reads.

The field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \alpha^{-1} T_{\mu\nu}, \quad (3)$$

$$D_\mu D^\mu \phi + \frac{\partial U(\phi)}{\partial \phi} + \bar{\psi} \frac{\partial f(\phi)}{\partial \phi} \psi = \frac{\partial \alpha(\phi)}{\partial \phi} R, \quad (4)$$

$$i \Gamma^\mu \nabla_\mu \psi - (m + f(\phi)) \psi = \frac{\partial V(\bar{\psi}\psi)}{\partial \bar{\psi}}, \quad (5)$$

$$i \nabla_\mu \bar{\psi} \Gamma^\mu + (m + f(\phi)) \bar{\psi} = -\frac{\partial V(\bar{\psi}\psi)}{\partial \psi}, \quad (6)$$

where $T_{\mu\nu} = T^{\phi}_{\mu\nu} + T^{D}_{\mu\nu} + T^{int}_{\mu\nu}$ and

$$T^{\phi}_{\mu\nu} = -\partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi - g_{\mu\nu} U(\phi) - g_{\mu\nu} \Box \alpha(\phi) + \partial_\mu \partial_\nu \alpha(\phi),$$

$$T^{D}_{\mu\nu} = \frac{i}{4} \left[\bar{\psi} \Gamma^\mu \nabla^\nu \psi + \bar{\psi} \Gamma^\nu \nabla^\mu \psi - \nabla^\nu \bar{\psi} \Gamma^\mu \psi - \nabla^\mu \bar{\psi} \Gamma^\nu \psi \right] - g_{\mu\nu} \mathcal{L}_D,$$

$$T^{int}_{\mu\nu} = -g_{\mu\nu} \bar{\psi} f(\phi) \psi,$$

A FLRW flat universe will be considered

$$ds^2 = dt^2 - a(t)^2[dx^2 + dy^2 + dz^2], \tag{7}$$

the tetrad, Dirac matrices and Spin connection reads

$$e_0^{\mu} = \delta_0^{\mu} \ , \ e_i^{\mu} = \frac{1}{a(t)}\delta_i^{\mu}, \tag{8}$$

$$\Gamma^0 = \gamma^0 \ , \ \Gamma^i = \frac{1}{a(t)}\gamma^i, \tag{9}$$

$$\Omega_0 = 0 \ , \ \Omega_i = \frac{1}{2}\dot{a}(t)\gamma_i\gamma_0, \tag{10}$$

where a dot for the time derivative have been introduced.

We consider now that the fields are homogenous and isotropic based on the observational fact that on a cosmological scale higher than 300Mpc the fields appear to be independent of the spatial coordinates in a post inflation evolution [1].

Fermion field equations (5) and (6) become

$$\dot{\psi} + \frac{3}{2}H\psi = -i(m + f(\phi))\gamma^0\psi - i\gamma^0\frac{\partial V}{\partial \bar{\psi}}, \qquad (11)$$

$$\dot{\bar{\psi}} + \frac{3}{2}H\bar{\psi} = i(m + f(\phi))\bar{\psi}\gamma^0 + i\frac{\partial V}{\partial \psi}\gamma^0 \ , \qquad (12)$$

where $H = H(t) = \frac{\dot{a}(t)}{a(t)}$.

$$\frac{d}{dt}\left(\bar{\psi}\psi\right) + 3H\left(\bar{\psi}\psi\right) = i\left(\frac{\partial V}{\partial \psi}\gamma^0\psi - \bar{\psi}\gamma^0\frac{\partial V}{\partial \bar{\psi}}\right). \qquad (13)$$

if $V\left(\bar{\psi}, \psi\right) = V\left(\bar{\psi}\Gamma\psi\right)$ then

$$S = S_0\left(\frac{a_0}{a}\right)^3, \qquad (14)$$

where $S = \bar{\psi}\psi$ has been defined.

Assuming $\omega_\psi = 0$, i.e. a dust type solution, equations (3 - 6) take the following form

$$\dot{\rho}_\phi + 3H\rho_\phi (1 + \omega_\phi) = -\frac{2\xi\phi\dot{\phi}(\rho_\phi + \rho_\psi)}{1 - \xi\phi^2} - Q, \quad (15)$$

$$\dot{\rho}_\psi + 3H\rho_\psi = Q, \quad (16)$$

$$Q = \beta S_0 a^{-3}\dot{\phi}, \quad (17)$$

Equation (17) is easily obtained by considering the Yukawa type of interaction $f(\phi) = \beta\phi$ and equation (14).
The last two equations can be combined to yield

$$\frac{d}{da}\left(\frac{\rho_\psi a^3}{\beta S_0} - \phi\right) = 0,$$

$$\phi = \frac{\rho_\psi}{\beta S_0} a^3 + c. \quad (18)$$

where the integration constant c will be shown to be zero.

Singularity point $a = a_c$ when $1 - \xi\phi^2(a_c) = 0$ or equivalently when $\rho_\psi(a_c) = \frac{\beta S_0}{\sqrt{\xi}}\frac{1}{a_c^3}$ with $\xi > 0$, implies validity only for $|\phi| < \phi(a_c)$ or $|\phi| > \phi(a_c)$. Equations can be symmetrized in the following form:

$$\frac{\ddot{a}}{a} = -\frac{1}{6}[\hat{\rho}_\phi + \hat{\rho}_\psi + 3(\hat{p}_\phi + \hat{p}_\psi)], \tag{19}$$

$$3H^2 = \hat{\rho}_\phi + \hat{\rho}_\psi, \tag{20}$$

through the re-definition of the densities and pressures to

$$\hat{\rho}_\phi = \alpha^{-1}\left(\frac{1}{2}\dot{\phi}^2 + U(\phi) + 6\xi H\phi\dot{\phi}\right), \tag{21}$$

$$\hat{\rho}_\psi = \alpha^{-1}\left(\bar{\psi}(m + f(\phi))\psi + V(\bar{\psi},\psi)\right), \tag{22}$$

$$\hat{p}_\phi = \alpha^{-1}\left(\frac{1}{2}\dot{\phi}^2 - U(\phi) - 2\xi\left(\phi\ddot{\phi} + \dot{\phi}^2 + 2H\phi\dot{\phi}\right)\right), \tag{23}$$

$$\hat{p}_\psi = \alpha^{-1}\left(\frac{1}{2}\bar{\psi}\frac{\partial V}{\partial\bar{\psi}} + \frac{1}{2}\frac{\partial V}{\partial\psi}\psi - V(\bar{\psi},\psi)\right). \tag{24}$$

We also have the following equation

$$\hat{\rho}_\phi\left(1+\omega_\phi\right)=\alpha^{-1}\{H^2\left(\phi'^2 a^2\left(1-2\xi\right)-2\xi\phi\phi'' a^2\right)-\xi\phi\phi' a^2\frac{d}{da}H^2\}, \quad (25)$$

which will be used in numerical simulation to find ϕ_0. The equations of state are

$$\begin{aligned}\hat{p}_\phi &= \omega_\phi\hat{\rho}_\phi,\\ \hat{p}_\psi &= \omega_\psi\hat{\rho}_\psi,\end{aligned}$$

which yields the following redefined conservation laws

$$\dot{\hat{\rho}}_\phi+3H\left(\hat{\rho}_\phi+\hat{p}_\phi\right) = -\frac{2\xi\phi\dot{\phi}}{1-\xi\phi^2}\hat{\rho}_\psi-\hat{Q}, \quad (26)$$

$$\dot{\hat{\rho}}_\psi+3H\left(\hat{\rho}_\psi+\hat{p}_\psi\right) = \frac{2\xi\phi\dot{\phi}}{1-\xi\phi^2}\hat{\rho}_\psi+\hat{Q}, \quad (27)$$

where $\hat{Q}=\alpha^{-1}\bar{\psi}\frac{\partial f}{\partial\phi}\psi\dot{\phi}$ has been defined.

Through these redefinitions the energy-momentum tensor is now conserved $\nabla^\mu \hat{T}_{\mu\nu} = 0$, with $\hat{T}_{\mu\nu} = \alpha^{-1} T_{\mu\nu}$.

At this point it is clear that we can interpret *fermionic fields as sources of dark matter* and *bosonic fields as sources of dark energy* as claimed above.

We see that positive acceleration imposes

$$\hat{\rho}_\phi + \hat{\rho}_\psi + 3\omega_\phi \hat{\rho}_\phi < 0, \tag{28}$$

from this equation we find $\omega_\phi < -\frac{1}{3}(1+r)$, in order to accelerated expansion to make sense, where we have defined $r = \frac{\hat{\rho}_\psi}{\hat{\rho}_\phi}$ called the coincidence parameter, with $r_0 \approx \frac{3}{7}$ the actual value.

$$\hat{Q} = 3\lambda H \hat{\rho}_\psi, \tag{29}$$

$$\hat{Q} = 3\lambda H \hat{\rho}_\phi, \tag{30}$$

where λ is a positive parameter (i.e. scalar field is transferring into dark matter fermions)
A dust type of model i.e. $\omega_\psi = 0$ or equivalently $p_\psi = 0$ implies

$$V\left(\bar{\psi}, \psi\right) = \frac{1}{2}\left(\bar{\psi}\frac{\partial V}{\partial \bar{\psi}} + \frac{\partial V}{\partial \psi}\psi\right). \tag{31}$$

Parity conservation on (31) implies

$$V\left(\bar{\psi}\psi\right) = V(S) = b\,S, \tag{32}$$

where b is an integration constant, replacing on (22)

$$V(S) = -mS - \beta c S, \tag{33}$$

Using (27) and (29) an analytical solution for $\rho_\psi(a)$ is found

$$\rho_\psi = \rho_{\psi_0} a^{3(\lambda-1)}, \tag{34}$$

with

$$\phi = \phi_0 a^{3\lambda}, \tag{35}$$

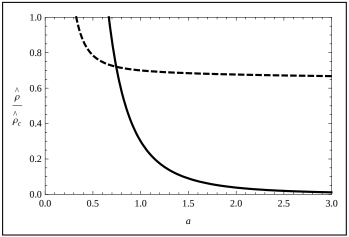
where we have identified $\phi_0 = \frac{\rho_{\psi_0}}{\beta S_0}$ and $a_0 = 1$. The λ dependence arises as a direct consequence of the interaction term. Replacing these results on (26) we obtain an equation for ρ_ϕ that can be solved numerically.

From (25) ϕ_0 turns out to be

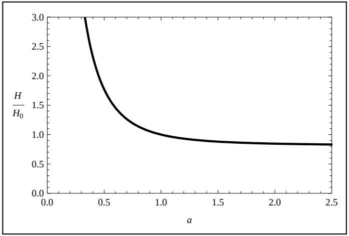
$$\phi_0 = \pm\sqrt{\frac{(1+\omega_0)\frac{\hat{\rho}_{\phi_0}}{\hat{\rho}_c}}{(3\lambda^2 - 12\xi\lambda^2 + 2\xi\lambda) + 3\xi\lambda\left(1+\omega_0\frac{\hat{\rho}_{\phi_0}}{\hat{\rho}_c}\right) + \xi(1+\omega_0)\frac{\hat{\rho}_{\phi_0}}{\hat{\rho}_c}}}, \tag{36}$$

where $\hat{\rho}_c = 3H_0^2$ is the density critical value at $t = t_0$.

Case I: $\hat{Q} = 3\lambda H \hat{\rho}_\psi$



(a)



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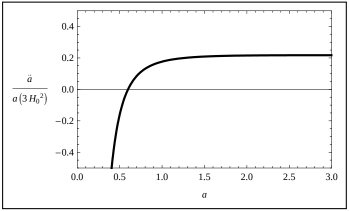

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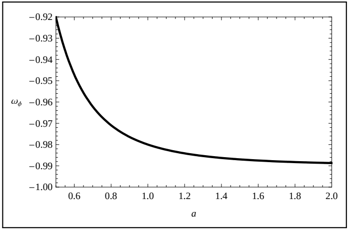
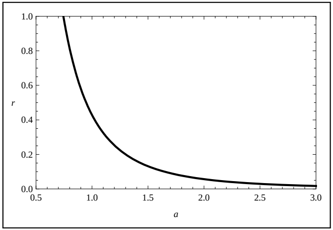


(d)

Case II: $\hat{Q} = 3\lambda H\hat{\rho}_\phi$



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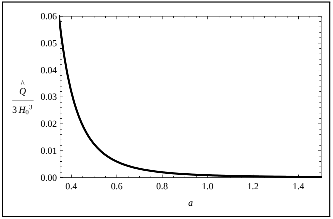

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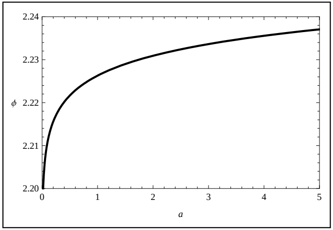

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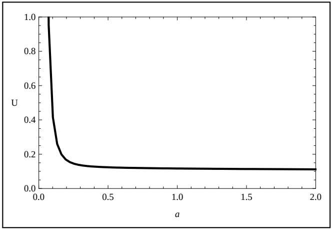
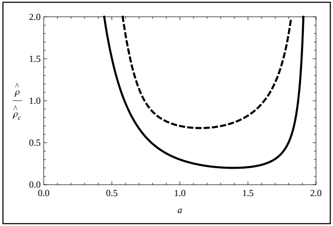


(h)

In this case the solutions are all numerical. Case II presents two branches for possible initial conditions for ϕ_0

$$\begin{aligned}\phi_0 &= 3{,}45 > \phi_c, \\ \phi_0 &= 2{,}43 < \phi_c.\end{aligned}$$

The first branch is similar behavior to Case I and is a quintessence universe. The second it has a phantom type of behavior that can cross into quintessence.



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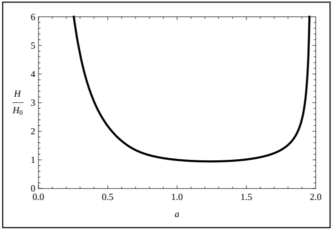
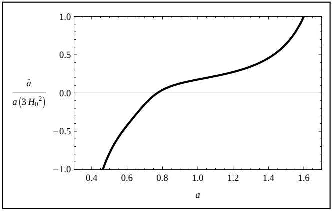


(j)

Case II: $\hat{Q} = 3\lambda H\hat{\rho}_\phi$



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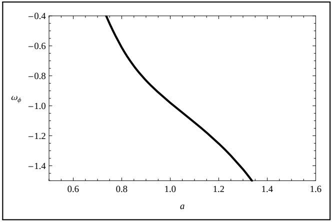
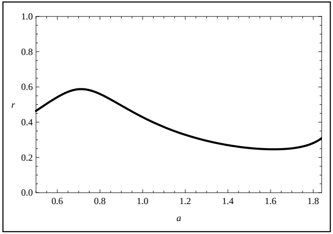
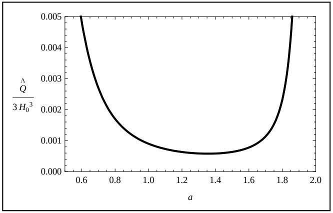


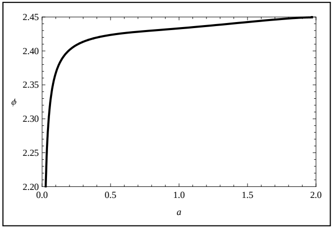
(l)



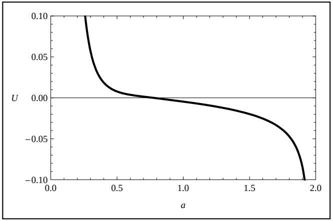
(m)



(n)



(ñ)



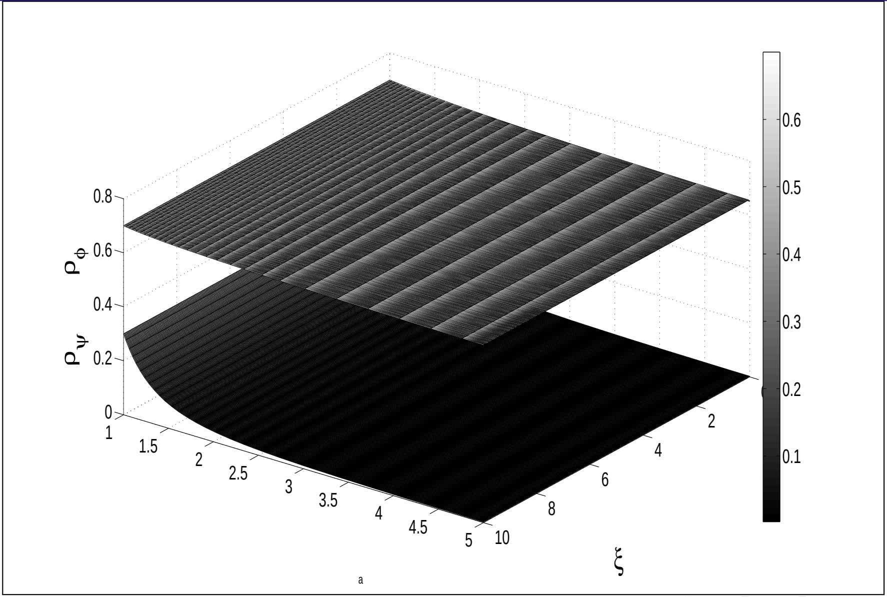
(o)

Minimal coupling does not allow to pass over the phantom barrier $\omega = -1$ for the initial conditions chosen. This seems to be the general behavior, reinforced by the exploration of the parameter space for large ranges. The behavior for this type of model in the future is quintessence.

In Case II, non-minimal coupling allows to pass over the $\omega = -1$ barrier for certain initial conditions, although the model seems to be very insensitive of the value of the coupling parameter ξ. This type of model delivers a type III cosmological singularity produced mainly because of the α factor.

The incorporation of the scalar field as source of dark energy and the fermionic field as source of dark matter seems to model very adequately the dynamics as well as the interaction in the cosmological models found.

In Case II, the ξ parameter does not take such a prominent effect on the energy density solutions.


0.8
0.6
0.4
0.2
0
ρ_ϕ
ρ_ψ
1
1.5
2
2.5
3
3.5
4
4.5
5
a
2
4
6
8
10
ξ
0.6
0.5
0.4
0.3
0.2
0.1

This ξ parameter insensibility can be take under account to find an almost-analytical solution in the case of cosmological constant universe.

$$\phi_{\xi}(t) = \sqrt{\frac{\theta_{\xi}}{2\pi\xi}} \left[\int_{0}^{t} \frac{dt'}{a(t')} \right]^{\frac{1}{2}}, \qquad (37)$$

where

$$\theta_{\xi} = \sum_{i=1}^{\infty} a_i \xi^i . \qquad (38)$$

The importance of this solution is that being valid in the barrier $\omega_{\phi} = -1$ permits the comparison of solutions on either side of the barrier, this let us propose the following remark.

Remark Let the solutions of a cosmological model derived from the Lagrangian (2) be valid in the interval $[a, b] \subseteq \mathbb{R}$, then a scalar field solution fulfilling

$$\phi(t) > \phi_{\xi}(t) \quad \forall\ t \in [a, b], \qquad (39)$$

belongs to a phantom type of universe, whereas a scalar field solution fulfilling

$$\phi\left(t\right) < \phi_{\xi}\left(t\right) \quad \forall\ t \in [a, b], \tag{40}$$

belongs to a quintessence type of universe
To complete the view, an almost-analytical solution is obtained for the Hubble parameter in terms of the scalar function

$$H = \frac{\beta S_0}{2} \int_{t_0}^{t} \frac{\phi_{\xi}\left(t'\right)}{\left(\xi \phi_{\xi}^{2}\left(t'\right) - 1\right) a'\left(t\right)^{3}} dt',$$

see appendix for details.
We will show here how to find a general Green's function for a non-minimal coupling at the boundary of cosmological constant. Using equations (21), (23), and the equations of state we have:

$$\rho_{\phi}\left(1 + \omega_{\phi}\right) = \left(\dot{\phi}^{2} - 2\xi L_{H}\left(\tfrac{1}{2}\phi^{2}\right)\right), \tag{41}$$

where the operator $L_H = \frac{d^2}{dt^2} + H(t)\frac{d}{dt}$ has been defined.
Note that on the left hand of equation (41), as ρ_ϕ is positive defined then for a cosmological constant solution $(\omega_\phi = -1)$ follows

$$\frac{1}{2\xi}\dot{\phi}_\xi^2 = L_H\left(\frac{1}{2}\phi_\xi^2\right). \tag{42}$$

The formal Green's function solution for the previous equation is

$$\phi_\xi^2 = \frac{1}{\xi}\int_0^\infty G(t,s)\left(\frac{d\phi_\xi}{ds}\right)^2 ds, \tag{43}$$

where the kernel $G : R\bigotimes R \to R$ fulfills $\mathfrak{L}_H G(t,s) = \delta(t-s)$ and $\mathfrak{L}_H = \frac{\partial^2}{\partial t^2} + H(t)\frac{\partial}{\partial t}$.
Let us define the following auxiliary function $K : R\bigotimes R \to R$: $K(t,s) = \frac{\partial}{\partial t}G(t,s)$ which fulfills the following differential equation

$$\frac{\partial}{\partial t}K(t,s) + H(t)K(t,s) = \delta(t-s), \tag{44}$$

by performing a Fourier Transform equation (44) can be written

$$\frac{\partial}{\partial t}K(t,x)+H(t)\,K(t,x)=\frac{1}{\sqrt{2\pi}}\exp(-ixt)\,. \tag{45}$$

The equation (45) is the Bernoulli's differential equation, which has a known solution

$$K(t',x)=\frac{1}{\sqrt{2\pi}}\frac{\int_0^{t'}\exp\left(\int_0^{t'''}H(t''')\,dt'''\right)\exp(-it''x)\,dt''}{\exp\left(\int_0^{t'}H(t'')\,dt''\right)}, \tag{46}$$

by using the Hubble's parameter and an inverse Fourier Transform it follows

$$K(t',s)=\frac{1}{2\pi a(t')}\int_0^{t'}a(t'')\,\delta(t''-s)\,dt''=\frac{1}{2\pi}\frac{a(s)}{a(t')}, \tag{47}$$

then the kernel is finally found to be

$$G\left(t,s\right)=\frac{a\left(s\right)}{2\pi}\int\limits_{0}^{t}\frac{dt'}{a\left(t'\right)}. \tag{48}$$

By replacing (48) on (43)

$$\phi_{\xi}^{2}\left(t\right)=\frac{1}{2\pi\xi}\int\limits_{0}^{t}\frac{dt'}{a\left(t'\right)}\left[\int\limits_{0}^{\infty}a\left(s\right)\left(\frac{d\phi_{\xi}}{ds}\right)^{2}ds\right],$$

note that the expression in brackets only depends on the parameter ξ which allows to write

$$\phi_{\xi}\left(t\right)=\sqrt{\frac{\theta_{\xi}}{2\pi\xi}}\left[\int\limits_{0}^{t}\frac{dt'}{a\left(t'\right)}\right]^{\frac{1}{2}},$$

which is equation (37), where $\theta_\xi = \int\limits_0^\infty a(s)\left(\frac{d\phi_\xi}{ds}\right)^2 ds$ has been defined.

Assuming θ_ξ is analytic then it must be of first order in ξ or higher to ensure convergence, as is readily seen on equation (41). A Taylor expansion is therefore of the form:

$$\theta_\xi = \sum_{i=1}^{\infty} a_i \xi^i,$$

which is the equation (38).

On the other hand, equation (25) can be written

$$\left(\rho_\phi + \rho_\psi\right)' + 3\frac{\rho_\psi}{a} + \frac{\xi}{\left(1 - \xi\phi_\xi^2(a)\right)}\left(\rho_\phi + \rho_\psi\right)\frac{d}{da}\left(\phi_\xi^2(a)\right) = 0,$$

which by aim of equation (18) takes the form

$$6HH'\left(1 - \xi\phi_\xi^2(a)\right) + 3\frac{\beta S_0}{a^4}\phi_\xi(a) = 0,$$

which finally can be written

$$H = \frac{\beta S_0}{2} \int_{t_0}^{t} \frac{\phi_\xi (t')}{\left(\xi \phi_\xi^2 (t') - 1\right) a' (t)^3} dt',$$

which is equation (41).

E. Komatsu, et al., Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation, Astrophys. J. Suppl. 192 (2011) 18. arXiv:1001.4538, doi:10.1088/0067-0049/192/2/18.