RÉNYI AND ENTANGLEMENT ENTROPIES:

some holographically calculable contributions

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High hopes: holography in AdS

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- (Dirac'36, "Wave equations in conformal space")
  conformal group in Minkowski$_d$ $\Leftrightarrow$ isometries of $(A)dS_{d+1}$
- (Fefferman-Graham'85, "Conformal invariants")
  conformal manifold $\mathcal{M}_d$ $\Leftrightarrow$ conformal infinity of Poincaré metric
- (Maldacena'97, 15 years of AdS/CFT correspondence)
  conformal field theory $\Leftrightarrow$ string/M-theory, (quantum) gravity

Outline

Perfect sense: extracts from AdS/CFT dictionary

- 't Hooft coupling ⇔ string length: $\lambda^{-1/4} = l_s$
- rank of the gauge group ⇔ Planck length : $N^{-1/4} = l_P$
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- energy scale ⇔ radial direction (IR-UV connection)
- trace anomaly ⇔ volume anomaly (Henningson+Skenderis'98)
- entanglement entropy ⇔ area of minimal surface (Ryu+Takayanagi'06)

A saucerful of secrets: entanglement

Entanglement: measure of “quantum-ness” (EPR paradox and Schrödinger’s cat, 1935...and today’s quantum information theory)

Entanglement/geometric entropy: von Neumann entropy of reduced density matrix

$$\rho_A = tr_B\{\rho_{A\oplus B}\} \qquad S_{EE} = -tr_A\{\rho_A \ln \rho_A\}$$

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Area law (Srednicki'93) : free fields in d spatial dimensions with spherical entangling surface

$$S_{EE} \sim \frac{\textit{Area of entangling surface}}{\epsilon^{d-1}}$$

Of interest in black hole physics: very reminiscent of Bekenstein-Hawking area law

Universal scaling at one-dimensional (d=1) conformal critical points (Holzhey+Larsen+Wilczek'94):
probably the most ubiquitous formula in last decade's literature

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- Need entry of AdS/CFT dictionary (Brown+Henneaux'86): central charge of the algebra of asymptotic diffs (PBH) in AdS_3

$$c = \frac{3\,L}{2\,G_N}$$

Obscured by clouds: EE for free fields, higher dims,... $\Leftrightarrow$???

Universality: for $d > 1$ the leading area term is reg-scheme-dependent

Conundrum: for even $d+1$, a universal (and conformally invariant) logarithmic term with Type-A trace anomaly coefficient a in front.

Established numerically (Lohmayer+Neuberger+Schwimmer+Theisen'09) as well as analytically (Dowker'10).

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Despite many efforts (Myers,Sinha,Casini,Huerta,Fursaev,...), no clear holographic picture as yet.

Wearing the inside out: a holographic derivation in two steps

- First (e.g. free scalar): EE as thermal entropy in $S^1 \times H^d$ (Casini+Huerta'10)

  via conformal maps reduces to the computation of 1-loop effective action for a conformal scalar, i.e., determinant of the conformal Laplacian (Yamabe operator, Y)

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  boundary $M = S^1 \times H^d$, bulk $X = H^{d+2}$

  evaluate for a special value of the mass of the bulk field

  determinant and trace anomaly $\Leftrightarrow$ 1-loop effective action for bulk duals

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A saucerful of secrets: Rényi entropy / q-entropies

Rényi vs. Shannon:

$$S_q^{(R)} = \frac{1}{1-q} \ln \, tr \rho^q \qquad\qquad S_1^{(R)} = S_{EE}$$

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Mutatis mutandis

- boundary $M = S^1 \times H^d$ with different radii, bulk $X = H^{d+2}$ with conical singularity
- use Sommerfeld formula to work out the Green function/ heat kernel
- example, $d = 1$:

$$S_q^{(R)} = \frac{c}{6} \cdot (1 + \frac{1}{q}) \cdot \ln \frac{\ell}{\epsilon}$$

What shall we do now: Outlook

- spinor: *mutatis mutandis* ✓; higher spins: ???
- connection with entropy of extremal black holes (Solodukhin)
- geometry of entangling surface and type-B trace anomaly: Wald or not Wald (Myers et al.)
- guess: finite contributions (Klebanov et al.) $\Leftrightarrow$ q-deformed Patterson-Selberg zeta (Floyd L. Williams)
- c-theorem, candidate for odd dimensions (Ryu+Takayanagi,Myers+Sinha,Casini+Huerta,...)