Dark Energy Models with Lagrange Multipliers Revisited

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The Universe
There are observational evidence to establish that there have been two accelerating expansion stages in the evolution of our universe.
The $\Lambda$CDM model

Action (Gravitational Interaction + Matter content)

$$S = \int (\mathcal{L}_G + \mathcal{L}_m) \, d^4x$$

Geometry (Symmetries: Homogeneity + Isotropy):
Friedmann-Robertson-Walker metric

$$ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2} + d\Omega^2 \right)$$

Energy - momentum Tensor: Perfect Fluid

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \text{diag}(\rho, -p, -p, -p)$$

Flat: $k = 0$
Open: $k = -1$
Close: $k = +1$

Rad.: $p = \rho/3$
Dust: $p = 0$
CC: $p = -\rho$
We do not know what are the constituent parts of dark components.
The $\Lambda$CDM model

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PROBLEMS

- Cosmological Constant
- Coincidence Problem
Models for dark energy include...

- More general gravity theory
- Anisotropic or inhomogeneous models
- Interaction between different components
- Viscous cosmological models
- Exotic Fluids: Chaplygin gas
Given that DM and DE manifest themselves so far only through their gravitational interaction, it is appealing to consider a unified version of the dark sector.
The Action

\[ S = \int d^4x \sqrt{-g} \left( -\frac{R}{2\kappa^2} + K(\psi, X) + \lambda \left( X - \frac{1}{2} \mu^2(\psi) \right) \right) \]

- Gravity
  - Ricci scalar

- Lagrange multiplier
  - Potential

- Matter Content
  - \( \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \psi \nabla_\beta \psi \)
The Equations of Motion

\[ G^\nu_\mu = \kappa^2 T^\nu_\mu; \quad \nabla_\nu T^\nu_\mu = 0 \quad \text{and} \quad X = \frac{1}{2} \mu^2(\psi) \]

Einstein Eq.  Conservation Eq.  Constraint

\[ T^\nu_\mu = \text{diag}(\rho, -p, -p, -p) \]

Perfect Fluid Form

It is assumed that the dynamics of our present universe is dominated by only one dark component:

\[ \rho = \mu^2(K_X + \lambda) - K \]

\[ p = K(\psi, \mu^2(\psi)) \]

\[ \omega = \frac{p}{\rho} \]

State parameter
Dynamical Equations

FRW metric:

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \]

We can recover the dynamics of the dark sector when \( K = X \)

\[
\begin{align*}
\omega' &= 3\omega \left( 1 + \omega - \epsilon \sqrt{\frac{-6\omega}{1 - k\chi}} \right) \\
\chi' &= \chi(1 + 3\omega) \\
\epsilon' &= -\epsilon^2(\Gamma - 1) \sqrt{\frac{-6\omega}{1 - k\chi}} \\
\end{align*}
\]

\[
\begin{align*}
\frac{\chi}{\rho a^2} &= \frac{3}{\epsilon} \left( \frac{\mu_{\psi}}{\mu} \right) \\
\Gamma &= \frac{\mu_{\psi}\mu}{\mu_{\psi}^2} \\
\end{align*}
\]

Primes denote derivative respect to log (a) and subscript derivatives respect to \( \psi \)
Nearly Flat Scalar Field Potential

We consider a nearly flat scalar field potential i.e.: $\epsilon \ll 1$ and nearly constant, then

$$
\begin{align*}
\omega' &= 3\omega \left( 1 + \omega - \sqrt{\frac{\omega}{\omega_f}} \frac{(1 + \omega_f)}{\sqrt{1 - k\chi}} \right) \\
\chi' &= \chi(1 + 3\omega)
\end{align*}
$$

Where $\epsilon = \frac{3}{2\sqrt{6}} \frac{1 + \omega_f}{\sqrt{-\omega_f}}$ and $\omega_f$ has to be close to $-1$

For a constant potential, $\epsilon = 0$ and $\omega_f = -1$, the critical points are all independent of the spatial curvature
Dynamical System Analysis

Critical Points

\[ \chi_c = 0 \text{ and } \omega_c = \omega_f \]

\[ \chi_c = 0 \text{ and } \omega_c = 0 \]

\[ \omega_c = -\frac{1}{3} \text{ and } \chi_c = \frac{10 + \frac{3}{\omega_f} + 3\omega_f}{4k} \quad k \neq 0 \]

This critical points are independent of the curvature of spatial sections.
Dynamical System Analysis
Stability of Critical Points

We study the dynamics in the vicinity of the critical points by calculating the eigenvalues of the matrix:

\[
\begin{pmatrix}
\delta \omega' \\
\delta \chi'
\end{pmatrix} =
\begin{pmatrix}
3 \left( 1 + 2\omega_c - \frac{3\sqrt{\omega_c / \omega_f (1+\omega_f)}}{2\sqrt{1-k\chi_c}} \right) & -\frac{3k\omega_c \sqrt{\omega_c / \omega_f (1+\omega_f)}}{2(1-k\chi_c)^{3/2}} \\
3\chi_c & 1 + 3\omega_c
\end{pmatrix}
\begin{pmatrix}
\delta \omega \\
\delta \chi
\end{pmatrix}
\]

- (0, 0) \quad \Rightarrow \quad Both are always positives \quad \text{Unstable}
- (0, \omega_f) \quad \Rightarrow \quad Both are negatives if \(\omega_f < -\frac{1}{3}\) \quad \text{Attractor}
- \left(-\frac{1}{3}, \frac{10 + \frac{3}{\omega_f} + 3\omega_f}{4k}\right) \quad \Rightarrow \quad \begin{align*}
\omega_f < 0 & \& \omega_f > -\frac{1}{3} \\
\omega_f < -\frac{1}{3} & \& \omega_f > -1 \\
\omega_f < -1 & \quad \text{Unstable}
\end{align*}
Dynamical System Analysis

Physically interesting Critical Points

\((\chi_c, \omega_c) = (0, 0)\)

\((\chi_c, \omega_c) = (0, \omega_f)\)

\(\chi_c \rightarrow 0 \Rightarrow a \rightarrow \infty \quad \text{Future}\)

or \(\rho \rightarrow \infty \quad \text{Past}\)
Numerical Results

\[ \omega \]

\[ \log[a/a_0] \]

\[ \omega_0 = -0.7 \]

\[ \omega_f = -0.9 \]

\[ \omega_f = -1 \]

\[ \omega_f = -1.1 \]
Numerical Results

\[ \omega_0 = -0.7 \]

\[ \Omega_\psi = \frac{\rho}{3 H^2} \]

\[ \Omega_k = -\frac{k}{a^2 H^2} \]

\[ \Omega^0_k = \pm 0.005 \]

\[ \Omega_\psi + \Omega_k = 1 \]
Numerical Results

\[ \omega_0 = -0.7 \text{ and } \Omega_{\psi}^0 = 0.995 \]

Crossing the phantom divide

Big Rip
Bayesian Analysis

We numerically solve the set of equations for 3 free parameters:

\[
\begin{align*}
\omega'(N) &= 3 \left( 1 + \omega(N) - \sqrt{\frac{\omega(N)}{\omega_f}} \frac{1 + \omega_f}{\sqrt{1 + (\Omega_k^0 \xi(N))}} \right) \\
\xi'(N) &= \xi(N)(1 + 3\omega(N)) \\
\omega(0) &= \omega_0 \\
\xi(0) &= 1 - \Omega_k^0 \\
N &= \log[a/a_0] \text{ and } \xi \leftrightarrow \chi
\end{align*}
\]

We compute the \( \chi^2 \) function associated and minimize it:

\[
\chi^2 = \sum_{i}^{ndat} \left( \frac{f[\omega_0, \omega_f, \Omega_k^0] - f_0}{\sigma_f} \right)^2
\]
Supernova Ia Data

Union 2 compilation
557 SnIa
$0.01 < z < 1.39$
SnIa data without curvature
H(z) data without curvature
SnIa data with curvature
Results and Perspectives

- The dynamical system analysis of the model is independent of the curvature of spatial sections.

- We have used SnIa data to constraint the model with and without curvature. The analysis shows that current SnIa data does not rule out spatial curvature.

- The data seems to slightly favored a closed universe.

- Perspectives: It is possible to use this kind of model to describe early universe?

- Perspectives: To study structure formation in detail.